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DOMINATION EDGE INTEGRITY OF GRAPHS

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Abstract. In an analysis of the vulnerability of a communication network to disruption, the most important two questions that come to mind are (i) what is the number of elements that are not functioning and (ii) what is the size of the largest remaining group in which mutual communication still continues. Integrity is one of the well-known vulnerability measures interested in these questions. Depending on network models new vulnerability measures take a great role in any failure not only on nodes also on links which have special properties. Domination is an another famous concept in network design. Sundareswaran and Swaminathan introduced domination integrity such as $DI(G) = min\{\ |S| + m(G-S) : S \subset V(G)\}$ where $m(G-S)$ denotes the order of a largest component of graph $G-S$ and S is a dominating set of G. In this work we define a new measure edge domination integrity of a connected and undirected graph G such as $DI^{'}(G) = min\{\ |S| + m(G-S) : S \subseteq E(G)\}$ where $m(G-S)$ is the order of a maximum component of $G-S$ and S is an edge dominating set. In this paper we present some results concerning this parameter on graph structures P_n, C_n, $K_{m,n}$, $K_{1,n}$.

Keywords: Domination, integrity, edge domination, edge integrity, edge domination integrity.

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1 Introduction

A communication network can be considered to be highly vulnerable to disruption if there are failures on nodes or links. Any communication network can be modeled by a graph G where nodes are represented by vertices $V(G)$ and links are represented by edges such as $E(G)$. All graphs that we study in this paper are connected, undirected, do not contain loops and multiple edges. Connectivity or edge connectivity are simple measures of how easily a graph can be broken apart (Buckley & Harary, 1990). However, these measures are not enough to compare the stability of network designs with the same order. For example, if two different network structures with same order have same connectivity or edge connectivity value, then how anyone can say that one is more stable than the other. Connectivity also does not give any idea about what happens to graph after disruption, how many nodes or links are still in communication. Network designers are also interested in what happens in the remaining part of the network after destruction, how many nodes or links are still connected to each other and what is the communication between remaining parts. These questions suggest the concept of the integrity and the edge integrity of a graph. Both types of integrity were introduced by Barefoot et al. (1987) and Goddard & Swart (1990) has great contributions. Integrity or edge integrity have been widely studied on specific families of graphs and combinations of graphs and relationships with other parameters and bounds (Bagga et al., 1992); (Dundar & Aytac, 2004); (Mamut & Vumar, 2007). Bagga et al. (1994) have presented many results about edge integrity in a survey article. The order of a graph G (that is, the number of vertices) will generally be denoted by n. For a real number x; $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x and

$\lceil x \rceil$ denotes the smallest integer greater than or equal to x. Here $m(G)$ denotes the maximum order (number of vertices) of a component in graph G, the vertex-integrity is defined as

$$I(G) = min\{|S| + m(G - S) : S \subset V(G)\}$$

and the edge-integrity as

$$I'(G) = min\{|S| + m(G - S) : S \subseteq E(G)\}.$$

(As usual, V and E denote the vertex- and edge-sets of G.)

Domination is another important concept in graph theory. A subset S of V is called a dominating set of G if every vertex not in S is adjacent to some vertex in S. The domination number $\gamma(G)$ (or γ for short) of G is the minimum cardinality taken over all dominating sets of G (Arumugam & Velammal, 1998).

Definition 1. *A subset X of E is called an edge dominating set of G if every edge not in X is adjacent to some edge in X (Mitchell & Hedetniemi (1977); Arumugam & Velammal (1998)).*

Definition 2. *The edge domination number $\gamma'(G)$ (or γ' for short) of G is the minimum cardinality taken over all edge dominating sets of G (Arumugam & Velammal, 1998).*

Mitchell and Hedetniemi (1977) have introduced the concept of edge domination.

Definition 3. *The domination integrity of a connected graph G is denoted by $DI(G)$ and defined as $DI(G) = \min\{|S| + m(G - S) \ : S \text{ is a dominating set } \}$ where $m(G - S)$ is the order of a maximum component of $G - S$ (Sundareswaran & Swaminathan, 2010a).*

Definition 4. *A subset S of $V(G)$ is a DI-set if $DI(G) = \min\{|S| + m(G - S) \ : S \subset V(G)\}$ where S is a dominating set of G (Sundareswaran, 2010).*

Many new results domination integrity were found by Sundareswaran and Swaminathan (2010b, 2012). Vaidya and Kothari (2012) have discussed domination integrity in the context of some graph operations. The domination integrity of splitting graph of path P_n and cycle C_n was investigated by same authors (Vaidya & Kothari, 2013). Vaidya and Shah (2014a, 2014b) determined the domination integrity of total graphs of path P_n, cycle C_n and star $K_{1,n}$ and also determined the domination integrity of square graph of path. Computational complexity of domination integrity in graphs is studied by Sundareswaran and Swaminathan (2015).

2 Domination edge integrity

In this paper, we introduce the concept of domination edge integrity of a connected graph as a new vulnerability parameter.

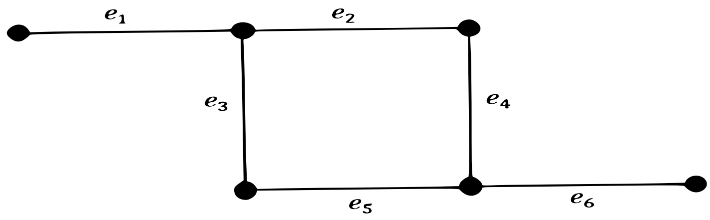
Definition 5. *The domination edge integrity of a connected graph G denoted by $DI^{'}(G)$ and defined by*

$$DI^{'}(G) = \min\{|S| + m(G - S) \ : S \text{ is an edge dominating set } \},$$

where $m(G - S)$ is the order of a maximum component of $G - S$.

Definition 6. *A subset S of $E(G)$ is a $DI^{'}$-set if $DI^{'}(G) = \min\{|S| + m(G - S) \ : S \subseteq E(G)\}$ where S is an edge dominating set of G.*

Example. Let's find the $DI^{'}$ value of the graph above G.
Let use an edge dominating set $S = \{e_2, e_5\} \subset E(G)$ where $|\mathrm{S}| = 2$ and $m(\mathrm{G} - \mathrm{S}) = 3$. Then we have $|S| + m(\mathrm{G} - \mathrm{S}) = 5$. Since there is no such an edge dominating set $X \subset E(\mathrm{G})$ which satisfies $|X| + m(G - X) < 5$ then $DI^{'}(G) = 5$.
Observation 1. For any graph G and $n \geq 2$, $2 \leq DI^{'}(G) \leq |E| + 1$.



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Proof. $DI^{'}$ takes its minimum value in graph K_2 since there is only one edge. To obtain components with minimum order in any connected graph G all edges must be removed from the graph. In other words edge dominating set should contain all the edges of the graph. In this case we obtain isolated vertices thus cardinality of $m\left(G-S\right)$ is 1 and minimum. Hence $DI^{'}=|E|+1$. The equality holds also for K_3, P_3. But there is no need to obtain components with size 1. The value $DI^{'}$ can be minimum due to the number of edges removed and the structure of remaining components. For example let us consider $DI^{'}(P_4)$. When we remove all the edges in P_4 we obtain isolated vertices and $m\left(G-S\right)=1$ and $|S|=4-1=3$. Hence $DI^{'}\left(P_4\right)=1+3=4=|E|+1$. But there is such an edge dominating set S that after removal of edge of this set from P_4, we obtain components with size 2. And $DI^{'}\left(P_4\right)=1+2=3$ which is less than $|E|+1$.
Hence $2\leq DI^{'}\left(G\right)\leq|E|+1$ holds. $|E|+1$ is a maximum upper bound value for $DI^{'}$. $\square$

Observation 2. $I^{'}(G)\leq DI^{'}\left(G\right)$.

Theorem 1. $DI^{'}\left(P_n\right)=\begin{cases} 2, & n=2, \\ \left\lceil\frac{n-1}{3}\right\rceil+\left\lceil\frac{n}{\lceil\frac{n-1}{3}\rceil+1}\right\rceil, & n\geq 3. \end{cases}$

Proof. The result is obvious for P_2. Let $n\geq 3$ and $S\subseteq E\left(P_n\right)$ be an edge dominating set. For any connected graph G, $m\left(G-S\right)\geq\frac{n}{\omega(G-S)}$ where $|S|=r$. (n is the order of G and $\omega(G-S)$ is the number of components of $G-S$). If $|S|=r$ edges are removed from P_n then $m\left(P_n-S\right)\geq\frac{n}{r+1}$ since $\omega\left(P_n-S\right)=r+1$. Hence S can be regarded as an edge dominating set. Since in P_n path graph, any edge can dominate at most 3 edges with itself, inequality can be written as $\geq\frac{n-1}{3}$. Thus we can write

$$\begin{aligned} DI^{'}\left(P_n\right) &= \min\left\{|S|+m\left(P_n-S\right): S \textit{ is an edge dominating set }\right\} \\ &\geq \min\left\{r+\frac{n}{r+1}\right\} \\ &\geq \min\left\{\left\lceil\frac{n-1}{3}\right\rceil+\left\lceil\frac{n}{\lceil\frac{n-1}{3}\rceil+1}\right\rceil\right\} \\ &\geq \left\lceil\frac{n-1}{3}\right\rceil+\left\lceil\frac{n}{\lceil\frac{n-1}{3}\rceil+1}\right\rceil. \end{aligned}$$

We can choose such a $S\subseteq E(P_n)$ edge dominating set that where $|S|=r=\left\lceil\frac{n-1}{3}\right\rceil$ and the order of the remaining components is $\left\lceil\frac{n}{\lceil\frac{n-1}{3}\rceil+1}\right\rceil$ at most . Due to edge domination and integrity concept $DI^{'}$ value must be an integer value so ceil function is used in the formula. Thus inequality becomes equality, proof is done. $\square$

Proposition 1. $\gamma^{'}\left(C_n\right)=\left\lceil\frac{n}{3}\right\rceil$, *for* $n\geq 3$ *(Arumugam and Velammal, 1998).*

Theorem 2. $DI^{'}\left(C_n\right)=\begin{cases} 4, & n=3,4, \\ \left\lceil\frac{n}{3}\right\rceil+3, & n\geq 5. \end{cases}$

Proof. For $n = 3$ and $n = 4$, it is obvious that $DI^{'}(C_n) = 4$. Let $n \geq 5$ and $S \subseteq E(C_n)$ be an edge dominating set. If $|S| = r$ edges are removed from C_n then $m(C_n - S) \geq \frac{n}{r}$ since $\omega(C_n - S) = r$. Hence S is chosen as an edge dominating set so from proposition 2.6 r should be given as an inequality $r \geq \lceil \frac{n}{3} \rceil$. Thus we can write
$DI^{'}(C_n) = \min\{|S| + m(C_n - S) \ : S \textit{ is an edge dominating set } \}$

$$\geq \min\left\{r + \frac{n}{r} \quad\right\}$$

$$\geq \min\left\{\frac{n}{3} + \frac{n}{\frac{n}{3}} \quad\right\}$$

$$\geq \frac{n}{3} + 3.$$

We can choose such a $S \subseteq E(C_n)$ edge dominating set that where $|S| = r = \lceil \frac{n}{3} \rceil$ and $m(C_n - S) = 3$ is satisfied for this set. Thus inequality becomes equality, proof is done. □

Theorem 3. *For $m \geq 2$ and $n \geq 3$, $DI^{'}(K_{m,n}) = min\{m, n\} + m + n - 1$.*

Proof. Let $K_{m,n}$ be a complete bipartite graph and v be a vertex whose degree is $min\{m, n\}$. The edges which is connected to v dominates any other edges in $E(K_{m,n})$. So S is the minimum edge dominating set of $K_{m,n}$. There is no such an edge dominating set whose cardinality is less than cardinality of S. If we remove edges of vertex v then $K_{m,n} - S$ includes an isolated vertex and a connected component whose size is $m + n - 1$. Therefore, $m(K_{m,n} - S) = m + n - 1$. If S is any edge dominating set of $K_{m,n}$ different from minimum edge dominating set, then $\left|S^{'}\right| + m\left(K_{m,n} - S^{'}\right) \geq min\{m, n\} + m + n - 1$. Hence, the result $DI^{'}(K_{m,n}) = min\{m, n\} + m + n - 1$. □

Theorem 4. $DI^{'}(K_{1,n}) = n + 1$.

Proof. Let $K_{1,n}$ be a star graph. Theorem 4 follow from Theorem 3. □

Theorem 5. *Let G be any graph and $e \in E(G)$, $DI^{'}(G - e) \geq DI^{'}(G) - 1$.*

Proof. Let S be an $DI^{'}$-set of $G-e$. Then S is an edge dominating set of $G-e$ and $DI^{'}(G - e) = |S| + m((G - e) - S)$. There are two cases of adding an edge;
Case 1 : e can be added between two vertices which are the vertices of an edge in S.
Case 2 : e can be added between two vertices which are not the vertices of any edge in S.
In both cases, $X = S \cup \{e\}$ is an edge dominating set for G. Then $|X| = |S| + 1$ and $m(G - X) = m((G - e) - S)$. Thus, $DI^{'}(G) \leq |X| + m(G - X) = |S| + m((G - e) - S) + 1 = DI^{'}(G - e) + 1$. As a result, $DI^{'}(G - e) \geq DI^{'}(G) - 1$. □

3 Conclusion

In this paper we introduced a new vulnerability measure domination edge integrity of graphs and computed results for P_n, C_n, $K_{m,n}$, $K_{1,n}$ which are commonly used network models. We also found lower and upper bounds for domination edge integrity value of a graph.

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